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Difference Equations over p -adic Fields

M. VAN DER PUT

Introduction and Summary

The starting point of this work is the difference equation

$$a_1 f(x + b_1) + \cdots + a_s f(x + b_s) = g(x), \quad (0.1)$$

where f and g are continuous functions of \mathbb{Z}_p^u into a field $K \supset \mathbb{Q}_p$; $x, b_1, \dots, b_s \in \mathbb{Z}_p^u$; $a_1, \dots, a_s \in K$.

In Section 1, a general definition of difference equation and difference operator is given for the case of continuous functions of a compact zero-dimensional group G into a complete non-archimedean valued field K . The algebra of all difference operators $W(G, K)$, belonging to G and K , turns out to be a commutative Banach algebra over K . In the case $G = \mathbb{Z}_p^u$ and $K \supset \mathbb{Q}_p$ or $\text{char } K = p$, the algebra $W(G, K)$ is especially nice, since it is isomorphic to the ring $K\langle X_1, \dots, X_u \rangle$ of all power series over K in the variables X_1, \dots, X_u , converging and bounded on $\{(a_1, \dots, a_u) \in K_{\text{alg}}^u \mid |a_i| < 1\}$, where K_{alg} is an algebraic closure of K .

In Section 2 it is shown that, provided that the valuation of K is discrete, $K\langle X_1, \dots, X_u \rangle$ shares many properties with the affinoid algebras over K . The algebraic exploration of $K\langle X_1, \dots, X_u \rangle$ leads in Section 3 to a detailed description of the solution of difference equations (Theorem (3.1)) and in Section 4 to Theorem (4.3) which states that, provided that the valuation of K is discrete, there is a 1–1 correspondence between the set of all closed, translation invariant, subspaces of the Banach space of all continuous functions of \mathbb{Z}_p^u into K , and the set of all ideals of the noetherian ring $K\langle X_1, \dots, X_u \rangle$.

1. The General Form of a Difference Equation

Let G be a compact zero-dimensional group and K be any complete non-archimedean valued field. We will use the following notations:

$C(G) = C(G \rightarrow K)$ is the Banach algebra of all continuous functions $f: G \rightarrow K$. The norm is defined by $\|f\| = \sup \{|f(x)| \mid x \in G\}$.

$M(G) = M(G, K)$ is the dual of $C(G)$, consisting of all bounded linear maps $\mu: C(G) \rightarrow K$, called K -valued measures on G , provided with the

usual norm $\|\mu\| = \sup \{ \|\mu(f)\| \mid f \in C(G), \|f\| \leq 1 \}$. The Banach space $M(G)$ is made into a Banach algebra by means of the convolution:

(1.1) $(\mu * \nu)(f) = \mu \hat{\otimes} \nu(F)$, where $\mu, \nu \in M(G)$, $f \in C(G)$, $F \in C(G \times G)$ is given by $F(x, y) = f(xy)$. Using ([2], p. 417) the fact that $C(G) \hat{\otimes} C(G) = C(G \times G)$ one easily checks the following: the formula (1.1) makes sense, $\|\mu * \nu\| \leq \|\mu\| \|\nu\|$; $*$ is bilinear, associative and in case G is commutative, $*$ is also commutative.

$W(G) = W(G, K)$ denotes the closed subalgebra of the Banach algebra of all bounded linear operators of $C(G)$ into $C(G)$, consisting of those operators which commute with all left-translations on $C(G)$.

(1.2) Theorem. *The Banach algebras $M(G)$ and $W(G)$ are isometrically isomorphic.*

Proof. Define $\vartheta: W(G) \rightarrow M(G)$ by $\vartheta(T) = T'(\delta_e)$, where $T \in W(G)$, $T': M(G) \rightarrow M(G)$ is the adjoint of T , e is the neutral element of G , $\delta_e \in M(G)$ is the measure given by $\delta_e(f) = f(e)$. Clearly ϑ is linear and $\|\vartheta\| \leq 1$.

Define $\eta: M(G) \rightarrow W(G)$ by $\eta(\mu)(\varphi)(a) = \mu(\tau_a \varphi)$, where $\mu \in M(G)$, $\varphi \in C(G)$, $\tau_a: C(G) \rightarrow C(G)$ is the left-translation given by $(\tau_a f)(x) = f(ax)$. The function $a \mapsto \mu(\tau_a \varphi)$ is continuous and thus belongs to $C(G)$. Moreover, for any $b \in G$ we have $\tau_b \eta(\mu)(\varphi) = \eta(\mu)(\tau_b \varphi)$. It follows that $\eta(\mu)$ is indeed an element of $W(G)$. Clearly $\|\eta\| \leq 1$.

A trivial checking yields: $\eta(\mu * \nu) = \eta(\mu) \eta(\nu)$, $\eta \vartheta = \text{id}$, $\vartheta \eta = \text{id}$. This completes the proof.

(1.3) Corollary. *For all $T \in W(G)$ and $\mu \in M(G)$, $T'(\mu) = \mu * \vartheta(T)$.*

Proof.

$\eta(\mu \circ T)(\varphi)(a) = \mu \circ T(\tau_a \varphi) = \mu(\tau_a T \varphi) = \eta(\mu)(T \varphi)(a) = (\eta(\mu) \circ T)(\varphi)(a)$, where $\varphi \in C(G)$, $a \in G$. Hence $\eta(T'(\mu)) = \eta(\mu \circ T) = \eta(\mu) T$. Applying ϑ we find $T'(\mu) = \mu * \vartheta(T)$.

Definition. The elements of $W(G)$ are called *difference operators*. A *difference equation* is an equation of the form $Tf = g$ with $T \in W(G)$.

Remarks. (1) The Eq. (0.1) is a difference equation.

(2) It is somewhat amazing that the condition " G has a K -valued Haar measure" is not needed in theorem (1.2). However the proof of (1.2) can be simplified under the assumption that G has a K -valued Haar measure by using the imbedding $C(G) \rightarrow M(G)$ induced by the Haar measure.

(3) Suppose that G is commutative, has a K -valued Haar measure and that K has enough roots of unity. (See [5] for the exact form of these conditions.)

Then difference equations are of a rather trivial nature. To see this we invoke [5], Theorem (5.2.10): "The set \hat{G} of continuous characters on G is an α -orthonormal base of $C(G)$, where $\alpha = |[G: 1]|$."

It is easy to see that $W(G)$ consists of all bounded linear maps on $C(G)$ which have the diagonal form with respect to the base \hat{G} of $C(G)$.

If the residue field of K has characteristic zero then every G has a Haar measure. A typical example of a commutative compact group which has no K -valued Haar measure, in case the residue characteristic of K is $p > 0$, is \mathbb{Z}_p the ring of p -adic integers. In the rest of this section we will calculate the algebras $M(\mathbb{Z}_p^u, K)$ for $u \in \mathbb{N}$. To do so we need the following definition:

Definition. Let K be any complete non-archimedean valued field (no restriction on the residue characteristic). Then $K\langle X_1, \dots, X_u \rangle$ denotes the set of all formal power series $\sum a_{\alpha_1, \dots, \alpha_u} X_1^{\alpha_1} \dots X_u^{\alpha_u}$ ($\sum a_{\alpha} X^{\alpha}$ for short) such that the set $\{a_{\alpha_1, \dots, \alpha_u}\}$ is bounded.

(1.4) Lemma. $R_u = K\langle X_1, \dots, X_u \rangle$ provided with the norm $\|\sum a_{\alpha} X^{\alpha}\| = \sup |a_{\alpha}|$ is a Banach algebra over K . Its norm is multiplicative (i.e. $\|fg\| = \|f\| \|g\|$).

Proof. Clearly R_u is a complete normed vector space over K . For $f, g \in R_u$ with $\|f\| \leq 1, \|g\| \leq 1$ it is immediate that all coefficients of fg have absolute value ≤ 1 . It follows that for any $f, g \in R_u$ we have $fg \in R_u$ and $\|fg\| \leq \|f\| \|g\|$. So R_u is a Banach algebra over K .

In showing that $\|fg\| = \|f\| \|g\|$ for all $f, g \in R_u$ we may suppose that $f \neq 0 \neq g$. Let $\varepsilon > 0$ and take a real number $\varrho, 0 < \varrho < 1$, such that $\|f\| \leq (1 + \varepsilon) \max |a_{\alpha}| \varrho^{|\alpha|}$ and $\|g\| \leq (1 + \varepsilon) \max |b_{\alpha}| \varrho^{|\alpha|}$. Here $f = \sum a_{\alpha} X^{\alpha}$, $g = \sum b_{\alpha} X^{\alpha}$, and $fg = \sum c_{\alpha} X^{\alpha}$. Let $a_{\alpha_0}, b_{\beta_0}$ be the first coefficients, in a fixed lexicographical ordering of \mathbb{N}_0^u (\mathbb{N}_0 denotes $\mathbb{N} \cup \{0\}$), satisfying $|a_{\alpha_0}| \varrho^{|\alpha_0|} = \max |a_{\alpha}| \varrho^{|\alpha|}$, $|b_{\beta_0}| \varrho^{|\beta_0|} = \max |b_{\alpha}| \varrho^{|\alpha|}$. Then we have $|c_{\alpha_0 + \beta_0}| \varrho^{|\alpha_0 + \beta_0|} = \left| \sum_{\alpha + \beta = \alpha_0 + \beta_0} a_{\alpha} b_{\beta} \right| \varrho^{|\alpha_0 + \beta_0|}$. In this sum $|a_{\alpha_0} b_{\beta_0}| \varrho^{|\alpha_0 + \beta_0|}$ is the only term with maximum value. Consequently $|c_{\alpha_0 + \beta_0}| \varrho^{|\alpha_0 + \beta_0|} = |a_{\alpha_0}| \varrho^{|\alpha_0|} |b_{\beta_0}| \varrho^{|\beta_0|}$. It follows that $\|fg\| \geq |c_{\alpha_0 + \beta_0}| \varrho^{|\alpha_0 + \beta_0|} \geq \max |a_{\alpha}| \varrho^{|\alpha|} \max |b_{\alpha}| \varrho^{|\alpha|} \geq (1 + \varepsilon)^{-2} \|f\| \|g\|$. This holds for all $\varepsilon > 0$, thus $\|fg\| = \|f\| \|g\|$.

Remark. The algebra $K\langle X_1, \dots, X_u \rangle$ has the affinoid algebra $K\{X_1, \dots, X_u\} = \{\sum a_{\alpha} X^{\alpha} \mid \lim a_{\alpha} = 0\}$ as a closed subalgebra. As we will see in Section 2, $K\langle X_1, \dots, X_u \rangle$ shares many properties (at least if the valuation of K is discrete) with $K\{X_1, \dots, X_u\}$. As a Banach space however $K\langle X_1, \dots, X_u \rangle$ is much larger than $K\{X_1, \dots, X_u\}$ since it is not of countable type and $K\{X_1, \dots, X_u\}$ is.

(1.5) Theorem The Banach algebra $M(\mathbb{Z}_p^u, K)$ is isometrically isomorphic to $K\langle X_1, \dots, X_u \rangle$ provided that the residue characteristic of K is p .

Proof. It is known that (see [2], p. 417), in case char $K = 0$ (so $K \supset \mathbb{Q}_p$), the set of polynomials $\left\{ \binom{t_1}{\alpha_1} \dots \binom{t_u}{\alpha_u} \mid (\alpha_1, \dots, \alpha_u) \in \mathbb{N}_0^u \right\}$, in the u variables

$(t_1, \dots, t_u) \in \mathbb{Z}_p^u$, form an orthonormal base of $C(\mathbb{Z}_p^u \rightarrow K)$. We adopt the notations: $t = (t_1, \dots, t_u)$; $\alpha = (\alpha_1, \dots, \alpha_u)$; $\binom{t}{\alpha} = \binom{t_1}{\alpha_1} \dots \binom{t_u}{\alpha_u}$. In the case that $\text{char } K = p$, the residues of $\binom{t}{\alpha}$ modulo $p\mathbb{Z}_p$ form again an orthonormal base of $C(\mathbb{Z}_p^u \rightarrow K)$. For convenience we denote the residues again by $\binom{t}{\alpha}$.

Now $\mu \in M(\mathbb{Z}_p^u, K)$ is determined by the bounded set $\left\{ \mu \left(\binom{t}{\alpha} \right) \mid \alpha \in \mathbb{N}_0^u \right\}$.

Any bounded set can occur and $\|\mu\| = \sup \left\{ \left\| \mu \left(\binom{t}{\alpha} \right) \right\| \mid \alpha \in \mathbb{N}_0^u \right\}$. It follows

that the map $\mathfrak{g}: M(\mathbb{Z}_p^u, K) \rightarrow K \langle X_1, \dots, X_u \rangle$ defined by $\mathfrak{g}(\mu) = \sum \mu \left(\binom{t}{\alpha} \right) X^\alpha$

is a linear isometry of Banach spaces. The formula $\binom{t+s}{\alpha} = \sum_{\beta+\gamma=\alpha} \binom{t}{\beta} \binom{s}{\gamma}$

implies that $(\mu * \nu) \left(\binom{t}{\alpha} \right) = \sum_{\beta+\gamma=\alpha} \mu \left(\binom{t}{\beta} \right) \nu \left(\binom{t}{\gamma} \right)$. By consequence $\mathfrak{g}(\mu * \nu) = \mathfrak{g}(\mu) \mathfrak{g}(\nu)$ and \mathfrak{g} is also a K -algebra homomorphism.

(1.6) Corollary. Suppose that the residue characteristic of K is $p > 0$. Then every difference operator $0 \neq T \in W(\mathbb{Z}_p^u, K)$ is surjective.

Proof. Combining (1.3), (1.4) and (1.5) one finds that the adjoint T' of T is injective and has a closed image in $M(\mathbb{Z}_p^u, K)$. Since the Banach space $C(\mathbb{Z}_p^u \rightarrow K)$ is of countable type over K , the corollary follows from the following lemma.

(1.7) Lemma. (a) Let E and F be Banach spaces over K . Suppose that for norm $\|\cdot\|$ on F , equivalent with the given one, the map $(F, \|\cdot\|) \rightarrow (F, \|\cdot\|)^*$ is an isometry into. Let $T: E \rightarrow F$ be a bounded linear map such that its adjoint $T': F' \rightarrow E'$ is injective and has a closed image. Then T is surjective.

(b) In either of the following cases, the condition imposed on F is satisfied:

- (1) F is of countable type over K and K is arbitrary.
- (2) F is arbitrary and K is spherically complete.

Proof. (a) Let $\varrho = \sup \{ |x| \mid x \in K, |x| < 1 \}$. Using the closed graph theorem for the operator T' one deduces the existence a constant $c > 0$ such that $c \|l\| \leq \|l \circ T\| \leq \|T\| \|l\|$ for all $l \in F'$. Let $E_0 = \{e \in E \mid \|e\| \leq 1\}$; $F_0 = \{f \in F \mid \|f\| \leq 1\}$; $\pi \in K, 0 < |\pi| < 1$; $\lambda \in K^*, |\lambda|^{-1} < \varrho c$. We claim that $F_0 \subseteq \lambda T(E_0) + \pi^2 F_0$.

Suppose not. The set $\lambda T(E_0) + \pi^2 F_0$ is convex, open, closed and contains the origin of F . The corresponding norm $\|\cdot\|$ on F is hence

equivalent with the original norm on F and satisfies:

$$\{f \in F \mid \|f\|^* < 1\} \subseteq \lambda T(E_0) + \pi^2 F_0 \subseteq \{f \in F \mid \|f\|^* \leq 1\}.$$

There is an element $x_0 \in F_0$ such that $x_0 \notin \lambda T(E_0) + \pi^2 F_0$. Hence $\|x_0\|^* \geq 1$. Since the map $(F, \|\cdot\|^*) \rightarrow (F, \|\cdot\|)$ is isometric there exists $l \in (F, \|\cdot\|)^*$ with $l(x_0) = 1$ and $\|l\|^* < qc|\lambda|$.

Then

$$\|l \circ T\| = |\lambda|^{-1} \|\lambda l \circ T\| \leq |\lambda|^{-1} q^{-1} \sup |l(\lambda T(E_0))| < |\lambda|^{-1} q^{-1} qc|\lambda| = c.$$

Further clearly $\|l\| \geq 1$. This yields the contradiction $c\|l\| > \|l \circ T\|$.

Now we will show that T is surjective. Take $f_0 \in F$. Using $F_0 \subseteq \lambda T(E_0) + \pi^2 F_0$ we find $e_0 \in E$ with $\|e_0\| \leq |\lambda| |\pi| \|f_0\|$ and $f_1 = f_0 - T(e_0)$ has norm $\leq |\pi| \|f_0\|$. Iterating, one obtains sequences $e_n \in E, f_n \in F$ with $\|e_n\| \leq |\pi|^{n-1} |\lambda| \|f_0\|, \|f_n\| \leq |\pi|^n \|f_0\|, f_{n+1} = f_n - T(e_n)$.

It follows that $f_0 = T\left(\sum_{i=0}^{\infty} e_i\right)$. Hence T is surjective.

(b) (1) If F is of countable type over K then there exists for every $\alpha, 0 < \alpha < 1$, an α -orthogonal base of F (see [3], Proposition (1.4)). It follows at once that $F \rightarrow F''$ is isometric.

(2) If K is spherically complete then the usual form of Hahn-Banach holds for all Banach spaces over K . Thus $F \rightarrow F''$ is isometric.

Remarks. (1) It is clear that the condition imposed on F is not superfluous, since there are Banach spaces F and E (both $\neq 0$) with $E' = F' = 0$.

(2) The analogue of (1.7) for real or complex Banach spaces is true. The proof of (1.7) can easily be translated to that case.

(3) As the proof of corollary (1.6) already indicates, a better knowledge of the structure of the algebra $K\langle X_1, \dots, X_u \rangle$ will give information on difference equations in \mathbb{Z}_p^n . In the next section we start the investigation of the algebras $K\langle X_1, \dots, X_u \rangle$.

(4) There is a more natural way to obtain the isomorphism of (1.5). For this purpose we introduce for a compact group G and a field K the set $\hat{G}(K)$ of all continuous characters of G into K .

For any field $L \supset K$ we have a canonical embedding $i: M(G, K) \hat{\otimes}_K L \rightarrow M(G, L)$. The "Fourier transform"

$$F: M(G, K) \rightarrow b(\hat{G}(L) \rightarrow L) = \{f: \hat{G}(L) \rightarrow L \mid f \text{ bounded}\}$$

by $F(\mu)(\chi) = i(\mu)(\chi); \mu \in M(G, K), \chi \in \hat{G}(L)$.

(1.8) Proposition. Suppose that the characteristic of the residue field of K is $p > 0$. Let $G = \mathbb{Z}_p^n$. Then:

(1) For any field $L \supset K$, $\hat{G}(L)$ is isomorphic to

$$\{(a_1, \dots, a_u) \in L^u \mid |a_i - 1| < 1\}.$$

In this isomorphism the element $(a_1, \dots, a_u) \in L^u$, all $|a_i - 1| < 1$, corresponds to the character $t = (t_1, \dots, t_u) \in \mathbb{Z}_p^u \mapsto a_1^{t_1} \dots a_u^{t_u} \in L$.

(2) $F: M(G, K) \rightarrow b(\hat{G}(L) \rightarrow L)$ is an isometry if the valuation of L is dense. Moreover $\text{im}(F)$ is the algebra of all bounded analytic functions on $O_u = \{(a_1, \dots, a_u) \in L^u \mid |a_i - 1| < 1 \text{ for all } i\}$ for which the power series expansion at $(1, \dots, 1)$ converges everywhere on O_u and has coefficients in K . This algebra is isomorphic to $K\langle X_1, \dots, X_u \rangle$.

Proof. (1) A continuous character $\chi: G \rightarrow L$ is determined by $\chi(0, \dots, 0, 1, 0, \dots, 0) = a_i \in L$. The continuity of χ implies $\lim_{n \rightarrow \infty} a_i^{p^n} = 0$. Hence we must have $|a_i - 1| < 1$ for all i . On the other hand, if $|a_i - 1| < 1$ for all i , then $t \mapsto a_1^{t_1} \dots a_u^{t_u}$ is a continuous character of G into L .

(2) Let $\mu \in M(G, K)$. Then

$$\begin{aligned} \mu(t \mapsto a_1^{t_1} \dots a_u^{t_u}) &= \mu\left(t \mapsto \sum_{\alpha} (a_1 - 1)^{\alpha_1} \dots (a_u - 1)^{\alpha_u} \binom{t}{\alpha}\right) \\ &= \sum_{\alpha} \mu\left(\binom{t}{\alpha}\right) (a_1 - 1)^{\alpha_1} \dots (a_u - 1)^{\alpha_u}. \end{aligned}$$

(Here we use the same notations as in the proof of (1.5)). So $F(\mu)$ has a power series expansion at $(1, \dots, 1)$ converging on O_u and with coefficients in K . It is also clear that every such power series is equal to a $F(\mu)$. If we put $X_i = (a_i - 1)$ we find that this algebra is isomorphic to $K\langle X_1, \dots, X_u \rangle$. The rest of the proposition follows from the well known fact that, in case the valuation of K is dense, the norm of any element $f \in K\langle X_1, \dots, X_u \rangle$ is equal to $\sup |f(O_u)|$.

2. Weierstrass Theorems for $K\langle X_1, \dots, X_u \rangle$

We start this section by showing that the rings $K\langle X_1, \dots, X_u \rangle$ are rather unpleasant if the valuation of K is dense.

(2.1) Proposition. Suppose that the valuation of K is dense then:

- (a) $K\langle X_1, \dots, X_u \rangle$ is not noetherian.
- (b) Not all ideals of $K\langle X_1, \dots, X_u \rangle$ are closed.
- (c) $K\langle X_1, \dots, X_u \rangle$ has maximal ideals of infinite codimension.

Proof. It is sufficient to give a proof for $u = 1$.

(a) Let (a_n) be a sequence in K such that $0 < |a_1| < |a_2| < \dots < |a_n| < 1$ for all n ; $\prod |a_n|^n = c > 0$. Consider $f_1 = \prod_{n=1}^{\infty} \left(1 - \left(\frac{X}{a_n}\right)^n\right) = \sum_{n=0}^{\infty} b_n X^n$. One sees that $\sup |b_n| = c^{-1} < \infty$. Hence f_1 and also all f_k , given by $\prod_{n=k}^{\infty} \left(1 - \left(\frac{X}{a_n}\right)^n\right)$ belong to $K\langle X \rangle$. An easy calculation yields that the

set of zeros of f_k in $\{a \in K_{\text{alg}} \mid |a| < 1\}$, where K_{alg} is an algebraic closure of K , is precisely $\{\omega_n a_n \mid \omega_n \text{ any } n\text{-th root of unity, } n \geq k\}$. Let J be the ideal in $K\langle X \rangle$ generated by all f_k .

If J would be finitely generated then J is generated by some f_i . But $f_{i+1} \notin (f_i)$. So J is not finitely generated and $K\langle X \rangle$ is not noetherian.

(b) It is known that (a) and (b) are equivalent (see [4], Theorem (1.4)).

(c) Let $M \supset J$ be a maximal ideal. If M has finite codimension then $K\langle X \rangle/M$ equals a finite field extension of K . The residue of X in that field is a root common to all f_k . This is a contradiction since the f_k 's have no common root in K_{alg} . Hence M has infinite codimension.

In the rest of this section we will suppose that the valuation of K is discrete. For any Banach algebra R over K its residue algebra $\{a \in R \mid \|a\| \leq 1\} / \{a \in R \mid \|a\| < 1\}$ is denoted by \bar{R} . The residue of an element $a \in R$, $\|a\| \leq 1$, is denoted by $\bar{a} \in \bar{R}$. In particular \bar{K} is the residue field of K which will also be denoted by k and any \bar{R} is a k -algebra. An elementary calculation shows that $\bar{R}_u = k[[X_1, \dots, X_u]]$ where $R_u = K\langle X_1, \dots, X_u \rangle$.

Definitions. An element $f \in R_u$, $\|f\| = 1$ is called *general* in X_u of degree s if its residue $\bar{f} \in k[[X_1, \dots, X_u]]$ has that property. More explicit:

$f = f(X_1, \dots, X_u) = \sum_{i=0}^{\infty} a_i(X_1, \dots, X_{u-1}) X_u^i$ should have the form $|a_i(0, \dots, 0)| < 1$ for $i = 0, \dots, s-1$ and $|a_s(0, \dots, 0)| = 1$.

A group A acting on R_u (as K -algebra automorphisms) is called *ample*, if there exists for every $f \in R_u$, $\|f\| = 1$ an element $a \in A$ with $a(f)$ general in X_u .

(2.2) Lemma. (a) Any K -algebra endomorphism of R_u is continuous and has norm 1. For all u -tuples $(a_1, \dots, a_u) \in R_u^u$ with $\|a_i\| \leq 1$, $|a_i(0)| < 1$ ($i = 1, \dots, u$) there exists one and only one K -algebra endomorphism of R_u mapping X_i onto a_i for all i . A K -algebra endomorphism of R_u is an automorphism if and only if the absolute value of its jacobian at 0 is 1.

(b) If $u = 1$ then $A = \{\text{id}\}$ is ample.

(c) If $\text{char } k = 0$ then the group $\text{Gl}_u(V_K)$, where $V_K = \{\lambda \in K \mid |\lambda| \leq 1\}$, acting as automorphisms on $K\langle X_1, \dots, X_u \rangle$ is ample.

(d) If $\text{char } k = p > 0$ then the action of the group $\text{Gl}_u(\mathbb{Z}_p)$ on R_u , defined by: if $a = (a_i)_{i,j=1}^u \in \text{Gl}_u(\mathbb{Z}_p)$ then $a((1 + X_i)) = (1 + X_i)^{a_{i,1}} \dots (1 + X_u)^{a_{i,u}}$, is ample.

Proof. (a) The set of maximal ideals of finite codimension of R_u is denoted by $\text{Max}_0(R_u)$. It is in 1-1 correspondence with the set $\{(a_1, \dots, a_u) \in K_{\text{alg}}^u \mid |a_i| < 1\}$, divided out by conjugacy under the Galois group of K_{alg} over K .

The well known fact $\|f\| = \sup \{|f(a_1, \dots, a_u)| \mid a_i \in K_{\text{alg}}, |a_i| < 1\}$ implies that $\|f\| = \sup \{|\hat{x}(f)| \mid \hat{x} \in \text{Max}_0(R_u)\}$.

Any K -algebra endomorphism φ of R_u induces a map $\varphi' : \text{Max}_0(R_u) \rightarrow \text{Max}_0(R_u)$ and hence $\|\varphi(f)\| \leq \|f\|$. In fact $\|\varphi\| = 1$ since $\varphi(1) = 1$.

Given $a_1, \dots, a_u \in R_u$, $\|a_i\| \leq 1$, $|a_i(0)| < 1$, the map $\varphi(f) = f(a_1, \dots, a_u)$ is well defined and has the required properties. Let ψ be another K -algebra endomorphism of R_u satisfying $\psi(X_i) = a_i$ for all i . Then the mappings $\varphi', \psi' : \text{Max}_0(R_u) \rightarrow \text{Max}_0(R_u)$ coincide, since φ and ψ coincide on $K[X_1, \dots, X_u]$, and all maximal ideals of finite codimension are generated by polynomials.

It follows that $\hat{x}(\varphi(f)) = \hat{x}(\psi(f))$ for all $\hat{x} \in \text{Max}_0(R_u)$. Hence

$$\|\varphi(f) - \psi(f)\| = \sup \{ |\hat{x}(\varphi(f)) - \hat{x}(\psi(f))| \mid \hat{x} \in \text{Max}_0(R_u) \} = 0.$$

So $\varphi(f) = \psi(f)$ for all f .

The Jacobian condition is clearly necessary. To prove sufficiency, we observe that φ induces for every ϱ , $0 < \varrho < 1$, an endomorphism φ_ϱ of $K\{X, \varrho\} = \{\sum a_\alpha X^\alpha \mid \lim a_\alpha \varrho^{|\alpha|} = 0\}$. It is well known ([1], Satz 2 on p. 409) that φ_ϱ has an inverse ψ_ϱ . For $0 < \varrho < \varrho' < 1$, $\psi_\varrho(X_i) = \psi_{\varrho'}(X_i)$ ($i = 1, \dots, u$). Therefore $a_i = \psi_\varrho(X_i) \in K\langle X_1, \dots, X_u \rangle$; $\|a_i\| \leq 1$, $|a_i(0)| < 1$, and the map ψ of R_u into itself, given by $\psi(X_i) = a_i$, for all i , is the inverse of φ .

(b) Obvious.

(c) It is sufficient to show that $Gl_u(V_K)$ working on the residue algebra $k[[X_1, \dots, X_u]]$ is ample. But that is well known (and easily checked) since $Gl_u(V_K)$ acts there as $Gl_u(k)$.

(d) Again it suffices to prove that the induced action of $Gl_u(\mathbb{Z}_p)$ on $k[[X_1, \dots, X_u]]$ is ample. For $\alpha_1, \dots, \alpha_{u-1} \in \mathbb{N}$ we consider the element $a \in Gl_u(\mathbb{Z}_p)$ given by $a(e_i) = e_i + p^{\alpha_i} e_u$, $i = 1, \dots, u-1$ and $a(e_u) = e_u$. The map $\varphi : k[[X_1, \dots, X_u]] \rightarrow k[[X_u]]$ defined by $\varphi(f) = a(f)(0, \dots, 0, X_u)$ can be calculated to be:

$\varphi(f) = f(X_u^{p^{\alpha_1}}, \dots, X_u^{p^{\alpha_{u-1}}}, X_u)$. We have to show that for any $f \in k[[X_1, \dots, X_u]]$, $f \neq 0$, there are $\alpha_1, \dots, \alpha_{u-1} \in \mathbb{N}$ such that $\varphi(f) \neq 0$.

The map φ may be decomposed as follows: $k[[X_1, \dots, X_u]] \xrightarrow{A^{z_1}} k[[X_2, \dots, X_u]] \rightarrow \dots \rightarrow k[[X_{u-1}, X_u]] \xrightarrow{A^{\alpha_{u-1}}} k[[X_u]]$, where $A^{z_i}(X_i) = X_u^{p^{z_i}}$, $A^{z_i}(X_j) = X_j$ for $j > i$. Hence it will suffice to show that for every $g_i \in k[[X_i, \dots, X_u]]$, $g_i \neq 0$, there exists $n \in \mathbb{N}$ with $A_n^i(g_i) \neq 0$.

Suppose $A_n^i(g_i) = 0$ then $g_i \in \ker A_n^i = (X_i - X_u^{p^n})$ which is a prime ideal of height one. Since $k[[X_1, \dots, X_u]]$ is noetherian, any $g_i \neq 0$ is contained in only a finite number of prime ideals of height one and we are done.

Remark. The assumption that K is discrete is rather hidden in the proof of (2.2). It is used in the formula $\hat{R}_u = k[[X_1, \dots, X_u]]$. However the proof of (a) is valid for any field K . In (b) and (c), (d) the assumption that the valuation of K is discrete is used. Indeed, if the valuation is dense then there are elements $f \in R_u$, $\|f\| = 1$, such that all coefficients of f have

absolute value < 1 . No automorphism can make an element of that type general in X_u .

(2.3) Theorem (Weierstrass Preparation and Division).

(1) Let $f \in R_u = K\langle X_1, \dots, X_u \rangle$, $\|f\| = 1$, be general in X_u of order s and let $g \in R_u$. Then " $g = qf + r$, $q \in R_u$, $r \in R_{u-1}[X_u]$ of degree $_{X_u}(r) < s$ " has a unique solution. Moreover the solution satisfies $\|g\| = \max(\|q\|, \|r\|)$.

(2) f can be written in the form $f = aw$, where $\|a\| = 1$, a is a unit and $w \in R_{u-1}[X_u]$ is a Weierstrass polynomial of degrees s . (i.e. $w = w_0(X_1, \dots, X_{u-1}) + w_1(X_1, \dots, X_{u-1})X_u + \dots + w_{s-1}(X_1, \dots, X_{u-1})X_u^{s-1} + X_u^s$, $\|w_i\| \leq 1$, $|w_i(0)| < 1$ for $i = 0, \dots, s-1$).

Proof. (1) We reduce the statement to the Weierstrass division theorem for $k[[X_1, \dots, X_u]]$. We may of course suppose $\|g\| \leq 1$. Let $\pi \in K$, $0 < |\pi| < 1$, be a uniformizing parameter. Then $\dot{g} = \dot{q}_0 \dot{f} + \dot{r}_0$ with $q_0, r_0 \in R_u$, $\|q_0\| \leq 1$, $\|r_0\| < 1$, $r_0 \in R_{u-1}[X_u]$ of degree $< s$. By induction one finds elements g_n, q_n, r_n all with norm $\leq |\pi|^n$; $g_n = g_{n-1} - q_n f - r_n$; $r_n \in R_{u-1}[X_u]$ of degree $< s$. It follows that $g = \sum_{n=0}^{\infty} q_n f + \left(\sum_{n=0}^{\infty} r_n \right)$.

So we found a solution (q, r) with $\max(\|q\|, \|r\|) = \|g\|$.

In showing uniqueness; let $0 = qf + r$ be a non trivial expression. We may suppose $\|q\| = \|r\| = 1$. Then $0 = \dot{q}\dot{f} + \dot{r}$ is a non trivial expression in $k[[X_1, \dots, X_u]]$. This is a contradiction.

(2) This follows in a formal way from (1): Put $X_u^s = qf + r$ and $w = X_u^s - r$ is a Weierstrass polynomial of degree s in X_u . Put $f = aw + r'$. Substituting one gets $f = aqf + r'$. Uniqueness of the division yields $aq = 1$ and $r' = 0$. Hence $f = aw$.

(2.4) Corollary. Let the group A act amply on R_u . Then: (1) For every ideal $I \subset R_u$ there exists a number d , $0 \leq d \leq u$ and an element $a \in A$ such that $K\langle X_1, \dots, X_d \rangle \cap a(I) = 0$ and R_u/I is a finite $K\langle X_1, \dots, X_d \rangle$ -module.

(2) R_u is noetherian; every ideal of R_u is closed; all maximal ideals of R_u have finite codimension over K ; R_u is a Jacobson ring.

Proof. We omit the proof, since it goes along the usual lines of function theory in several variables (see [1], Kap. I). The result: R_u is noetherian could have been obtained without the Weierstrass theorems as a simple application of [4]. Theorem (5.3).

For later use we insert the following results:

(2.5) Corollary. (1) $K\langle X \rangle$ is a principal ideal domain. Every ideal in $K\langle X \rangle$ is generated by a Weierstrass polynomial. For $T \in K\langle X \rangle$, $T \neq 0$, $\dim K\langle X \rangle / TK\langle X \rangle$ equals the number of roots of T in $\{a \in K_{\text{alg}} \mid |a| < 1\}$.

(2) Every ideal of R_u is an intersection of ideals which are of finite codimension in R_u .

(3) For every ideal of R_u which is of finite codimension, there are Weierstrass polynomials $p_1, \dots, p_u \in K\langle X \rangle$ such that all $p_i(X_i)$ belongs to the ideal.

Proof. (1) According to (2.3) part (2) every ideal $J \neq 0$ of $K\langle X \rangle$ contains some polynomial $\neq 0$. Let p be a polynomial in J of minimal degree. Again using (2.3) one sees that p must be a Weierstrass polynomial and generates J . The rest of (1) is obvious.

(2) Let J be an ideal of R_u and let J' be the intersection of all ideals containing J , which have finite codimension in R_u . Suppose that $J \neq J'$ and let $t \in J' \setminus J$. The ideal $J'' = \{a \in R_u \mid at \in J\}$ is proper and contained in some maximal ideal $M \subset R_u$. It follows from (2.4) that every $J + M^n$ has finite codimension in R_u . Hence $t \in \bigcap_{n=1}^{\infty} (J + M^n)$. Let S denote the localization of R_u at M . Since S is noetherian we have $JS = \bigcap_{n=1}^{\infty} (JS + (MS)^n)$ (see Zariski-Samuel, Part 1, Chapter 4, Section 7, Theorem 12').

It follows that $t \in JS$ and there exists $a \in R \setminus M$ with $at \in J$. This contradicts $J'' \subset M$.

(3) The kernel of the map $\varphi: K\langle X \rangle \rightarrow R_u/J$, given by $\varphi(X) = X_i$, is generated by some Weierstrass polynomial $p_i \neq 0$. This proves the statement.

3. Solution of Difference Equations on \mathbb{Z}_p^u

To avoid trivialities (see Section 1, Remark (3) after (1.3)), we suppose in this section that the characteristic of the residue field of K is $p > 0$. The group of continuous automorphisms of \mathbb{Z}_p^u is exactly $GL_u(\mathbb{Z}_p^u)$. The natural action of this group on $W(\mathbb{Z}_p^u, K) = M(\mathbb{Z}_p^u, K) = R_u$ is the action defined in Lemma (2.2), part (d). It follows that every difference operator $T, \|T\| = 1, T \in R_u$ equals a Weierstrass polynomial in X_u (of some order s), after a linear transform on \mathbb{Z}_p^u and after disregarding a unit of R_u . So without loss of generality we may suppose that T has the form:

$$T = w_0(\Delta_1, \dots, \Delta_{u-1}) \\ + w_1(\Delta_1, \dots, \Delta_{u-1}) \Delta_u + \dots + w_{s-1}(\Delta_1, \dots, \Delta_{u-1}) \Delta_u^{s-1} + \Delta_u^s,$$

where $\Delta_i = \tau_{e_i} - 1; e_1, \dots, e_u$ the base of \mathbb{Z}_p^u over \mathbb{Z}_p and $\|w_i\| \leq 1; |w_i(0, \dots, 0)| < 1$.

(3.1) Theorem. *With the notations above: (1) Given $g \in C(\mathbb{Z}_p^u)$; $f_0, \dots, f_{s-1} \in C(\mathbb{Z}_p^{u-1})$ there exists a unique $f \in C(\mathbb{Z}_p^u)$ satisfying $Tf = g$ and $f(t_1, \dots, t_{u-1}, i) = f_i(t_1, \dots, t_{u-1})$ for $i = 0, \dots, s-1$. In particular the kernel of T is isomorphic (as a Banach space) to $C(\mathbb{Z}_p^{u-1})^s$.*

(2) In case $u = 1$, $\ker T$ is finite dimensional and its dimension is equal to the number of zeros of T in $\{a \in K_{\text{alg}} \mid |a| < 1\}$.

If $a_1, \dots, a_s \in K_{\text{alg}}, |a_j| < 1$, are all the zeros of T , multiplicities n_1, \dots, n_s , then the set $\{t^i(1+a)^j \mid j=1, \dots, s; 0 \leq i < n_j\}$ is a base for $\ker T$.

Proof. (1) Put $f = \sum_{i=0}^{\infty} h_i \binom{t_u}{i}$; $h_i \in C(\mathbb{Z}_p^{u-1})$; $\lim \|h_i\| = 0$ and also $g = \sum_{i=0}^{\infty} g_i \binom{t_u}{i}$. The "boundary conditions" on f determine h_0, \dots, h_{s-1} .

The explicit form of $Tf = g$ now reads:

(3.2) $h_{i+s} + w_{s-1}(\Delta_1, \dots, \Delta_{u-1})h_{i+s-1} + \dots + w_0(\Delta_1, \dots, \Delta_{u-1})h_i = g_i$ for all $i \geq 0$.

So h_0, \dots, h_{s-1} determine all $h_i, i \geq s$. This implies the uniqueness of the solution f . All we have to show for the existence of a solution f is to prove that the formal solution $(h_i)_{i \geq 0}$ of (3.2) satisfies $\lim \|h_i\| = 0$.

To show this we introduce a function α on $C(\mathbb{Z}_p^{u-1})$. Let $\varrho, 0 < \varrho < 1$, be given. Then, if $h \in C(\mathbb{Z}_p^{u-1})$ has norm $< \varrho$ we put $\alpha(h) = 0$. If $\|h\| \geq \varrho$ then $\alpha(h) = \max \{|\alpha| + 1 \mid \alpha \in \mathbb{N}_0^{u-1} \text{ such that the coefficient of } \binom{t}{\alpha} \text{ in the expansion of } h \text{ has absolute value } \geq \varrho\}$.

Let N be such that $\|g_i\| < \varrho$ for all $i \geq N$. The properties of w_0, \dots, w_{s-1} imply that for $i \geq N$ we have

$$\alpha(h_{i+s}) < \max(\alpha(h_{i+s-1}), \dots, \alpha(h_i)).$$

Consequently there exists a number N' with $\|h_i\| < \varrho$ for all $i \geq N'$.

Hence $\lim \|h_i\| = 0$.

(2) The decomposition $T = aw$ of (2.3) and part (1) of (3.1) yields: $\dim \ker T = \dim \ker w = \text{degree } w = \text{number of zeros of } T \text{ in } \{a \in K_{\text{alg}} \mid |a| < 1\}$. It is further easy to verify that the given functions are linearly independent and belong to $\ker w = \ker T$.

Remark. As one can see from (3.1) the theory of difference equations on \mathbb{Z}_p^u is rather similar to the classical theory of differential equations with constant coefficients. A natural question is the following: "Can one extend (3.1) to difference equations with non-constant coefficients?"

The following proposition answers this in the negative, by showing that every bounded linear operator on $C(\mathbb{Z}_p^u)$ can be viewed as a difference operator with non-constant coefficients.

(3.3) Proposition. Every bounded K -linear map T of $C(\mathbb{Z}_p^u \rightarrow K)$ into itself can uniquely be written as $T = \sum g_{\alpha_1, \dots, \alpha_u} \Delta_1^{\alpha_1} \dots \Delta_u^{\alpha_u}$, $g_{\alpha} \in C(\mathbb{Z}_p^u \rightarrow K)$, and $\sup \|g_{\alpha}\| < \infty$. Moreover $\|T\| = \sup \|g_{\alpha}\|$.

Proof. A bounded linear operator T is determined by the bounded set $\left\{ T \left(\binom{t}{\alpha} \right) \mid \alpha \in \mathbb{N}_0^u \right\}$. We try to find $g_{\beta} \in C(\mathbb{Z}_p^u)$ with $G = \sum g_{\beta} \Delta^{\beta}$ has the property $G \left(\binom{t}{\alpha} \right) = T \left(\binom{t}{\alpha} \right)$ for all α .

Since $G\left(\begin{pmatrix} t \\ \alpha \end{pmatrix}\right) = \sum_{\beta \leq \alpha} g_{\beta} \begin{pmatrix} t \\ \alpha - \beta \end{pmatrix}$, there is a unique solution, and as is easily seen $\sup \|g_{\beta}\| = \sup \left\| T\left(\begin{pmatrix} t \\ \alpha \end{pmatrix}\right) \right\| = \|T\|$. Clearly $T = G$.

Remark. Another remarkable fact is that the special form of $M(\mathbb{Z}_p^u, K) = K\langle X_1, \dots, X_u \rangle$ has nothing to do with the groupstructure of \mathbb{Z}_p^u but is merely a phenomenon in linear operators. To illustrate this we give the following proposition:

(3.4) Proposition. *Let E be a Banach space over K (no restrictions on the characteristic of the residue field of K) having an orthonormal base $\{e_n | n \in \mathbb{N}_0\}$. The algebra of bounded linear operators on E is denoted by $L(E)$. For $A, B \in L(E)$ we put $[A, B] = AB - BA$. The operators $T, U \in L(E)$ are given by $T(e_n) = e_{n-1}$ for $n \geq 1$, $T(e_0) = 0$ and $U(e_n) = e_{n+1}$ for all $n \geq 0$. Then:*

(1) $\{A \in L(E) | [T, A] = 0\} = K\langle T \rangle$. Further for every $B \in L(E)$ there exists $A \in L(E)$ with $[T, A] = B$.

(2) $\{A \in L(E) | [U, A] = 0\} = K\{U\}$.

Proof. (1) If A commutes with T then there are elements $\lambda_0, \dots, \lambda_n, \dots \in K$ such that $A(e_n) = \lambda_0 e_n + \lambda_1 e_{n-1} + \dots + \lambda_n e_0$. Since $\sup \|A(e_n)\| = \|A\| < \infty$ it follows that $\sup |\lambda_n| = \|A\|$. Further clearly

$$A = \sum_{n=0}^{\infty} \lambda_n T^n \in K\langle T \rangle.$$

Given $B \in L(E)$ we define A on the base $\{e_n | n \in \mathbb{N}_0\}$ by induction using the following rule: $Ae_n = U(Be_n + ATe_n)$, $n \in \mathbb{N}_0$. Since $\|Ae_n\| \leq \|B\|$ for all n , A can be extended to a bounded linear operator on E . On the base (e_n) clearly $[T, A](e_n) = B(e_n)$. Hence $[T, A] = B$.

(2) If A commutes with U , A is determined by $Ae_0 = \sum_{i=0}^{\infty} \lambda_i e_i$, $\lim \lambda_i = 0$. So clearly $A = \sum_{i=0}^{\infty} \lambda_i U^i \in K\{U\}$.

4. Invariant Subspaces

In this section we suppose that G is a commutative compact zero-dimensional group. There are no restrictions on the field K . An *invariant subspace* of $C(G \rightarrow K)$ is a closed linear subspace of $C(G \rightarrow K)$ which is invariant under all translations $\{\tau_g | g \in G\}$.

(4.1) Lemma. *Let $A \subset B$ be invariant subspaces of $C(G \rightarrow K)$. (1) If $f \in A$ then $\Delta f \in C(G) \hat{\otimes} C(G)$, given by $\Delta f(x, y) = f(x + y)$, belongs to $A \hat{\otimes} A$.*

(2) *The dual A' of A is a commutative Banach algebra under the convolution defined by $(l * m)f = l \otimes m(\Delta f)$, where $f \in A$; $l, m \in A'$.*

(3) Let $W(A)$ denote the algebra of all bounded K -linear operators on A which commute with all translations on A . The mappings $\eta_A: A' \rightarrow W(A)$ and $\vartheta_A: W(A) \rightarrow A'$, given by $\eta_A(l)(f) = l \otimes 1_A(\Delta f)$; $\vartheta_A(T)(f) = T(f)(0)$; $l \in A'$, $T \in W(A)$ and $f \in C(G)$, are each others inverses. Moreover η_A and ϑ_A are isometrical K -Banach algebra isomorphisms.

(4) For any $T \in W(B)$, we have $T(A) \subseteq A$. Let $\sigma: W(B) \rightarrow W(A)$ denote the restriction map and let $\varrho: A \rightarrow B$ be the inclusion. Then the following diagram is commutative:

$$\begin{array}{ccc} B' & \xrightleftharpoons[\vartheta_B]{\eta_B} & W(B) \\ \varrho' \downarrow & & \downarrow \sigma \\ A' & \xrightleftharpoons[\vartheta_A]{\eta_A} & W(A) \end{array}$$

Proof. (1) For any $\varepsilon > 0$ there is an open subgroup $H \subset G$ such that all cosets $y_1 + H, \dots, y_s + H$ have the property $|f(a) - f(b)| \leq \varepsilon$ if $a, b \in y_i + H$. It follows that $\left\| \Delta f - \sum_{i=1}^s (\tau_{y_i}, f) \otimes \xi_{y_i + H} \right\| \leq \varepsilon$, where $\xi_{y_i + H}$ denotes the characteristic function of the open and closed set $y_i + H$. So $\Delta f \in A \hat{\otimes} C(G)$. Also $\Delta f \in C(G) \hat{\otimes} A$ and by consequence $\Delta f \in A \hat{\otimes} A$.

(2) Using (1) one sees that the formula in (2) makes sense. The checking that $*$ makes A' into a Banach algebra is easily done.

(3) By (1) and (2) one sees that η_A and ϑ_A are well defined. As in (1.2) one easily checks the properties of η_A and ϑ_A .

(4) It suffices to show that for $T \in W(B)$, $S = \eta_A \circ \varrho' \circ \vartheta_B(T) \in W(A)$ and $f \in A$ we have $T(f) = S(f)$. It follows then that $T(A) \subseteq A$ and $\sigma = \eta_A \circ \varrho' \circ \vartheta_B$. The other commutativity relations in the diagram follow at once. Now $S(f) = \eta_A \circ \varrho' \circ \vartheta_B(T)f = (\varrho' \circ \vartheta_B(T)) \otimes 1(\Delta f) = (\vartheta_B(T) \otimes 1)(\Delta f) = \eta_B \circ \vartheta_B(T)(f) = T(f)$ and we are done.

Definitions. For any ideal $J \subset W(G)$ we denote by $N(J)$ the set $\{f \in C(G) \mid Tf = 0 \text{ for all } T \in J\}$. For an invariant subspace $A \subset C(G)$ we put $I(A) = \{T \in W(G) \mid T(A) = 0\}$.

(4.2) Corollary. $N(J)$ is an invariant subspace of $C(G)$ and $I(A)$ is a closed ideal in $W(G)$. Suppose that either K is spherically complete or that G is metrizable. Then for any invariant subspace A we have $NI(A) = A$ and $W(G)/I(A)$ is isometrically isomorphic to the Banach algebra $W(A)$.

Proof. The first line of (4.2) is obvious. The condition imposed on K or G assures that we can use for subspaces E of $C(G)$ the following weak form of Hahn-Banach: "For every $l \in E'$ and $\varepsilon > 0$ there exists an extension $m \in C(G)'$ of l such that $\|m\| \leq (1 + \varepsilon) \|l\|$."

This is true since, in case K is spherically complete the usual form of Hahn-Banach holds and in case G is metrizable, $C(G)$ is a Banach space of countable type. (See the same argument in the proof of (1.7).)

The weak form of Hahn-Banach can be restated in the form: the map $C(G)/\{l \in C(G) \mid l(E) = 0\} \rightarrow E'$ is bijective and isometric.

Apply now (3) and (4) after substituting $B = C(G)$. Then one gets that $W(G)/I(A)$ is isometrically isomorphic to $W(A)$. Further, the adjoint i' of the inclusion map $i: A \rightarrow NI(A)$ has no kernel since $INI(A) = I(A)$. Again using the weak form of Hahn-Banach for subspaces of $C(G)$ we find that $A = NI(A)$.

Remark. (4.2) suggests the following question: "What ideals J of $W(G)$ satisfy $IN(J) = J$?" An answer to this question would give a description of the invariant subspaces of $C(G)$ since by (4.2) there is a 1-1 correspondence between the set of all invariant subspaces of $C(G)$ and the set of all ideals J of $W(G)$ satisfying $IN(J) = J$. (Under the same restriction as in (4.2).)

We will not pursue the question above for general groups G and fields K since that would lead us into harmonic analysis. However the case: $G = \mathbb{Z}_p^u$, the valuation of K discrete and the residue characteristic of K equal to p , is quite easily settled using the knowledge about $W(\mathbb{Z}_p^u, K)$ given in section 2.

(4.3) Theorem. *Let $G = \mathbb{Z}_p^u$ and let K be a field with a discrete valuation and with residue characteristic $p > 0$. Then: For every ideal J of $W(G, K) = K\langle X_1, \dots, X_u \rangle$ we have $IN(J) = J$ and consequently there is a 1-1 correspondence between the set of all ideals of $K\langle X_1, \dots, X_u \rangle$ and the set of all invariant subspaces of $C(\mathbb{Z}_p^u \rightarrow K)$.*

Proof. According to (4.2) and the remark following it we have only to show that $IN(J) = J$ for every ideal J of $K\langle X_1, \dots, X_u \rangle$. Using (2.5) part (2) one sees that it suffices to show that $IN(J) = J$ for all ideals J which are of finite codimension in $K\langle X_1, \dots, X_u \rangle$.

Let J be an ideal of finite codimension. Then $J \supseteq J_0 = (p_1(X_1), \dots, p_u(X_u))$ where $p_i \in K\langle X \rangle$ is a Weierstrass polynomial of degree r_i (see (2.5) part (3).) For $i = 1, \dots, u$, the kernel $A_i \subseteq C(\mathbb{Z}_p \rightarrow K)$ of $p_i(A)$ has dimension r_i , according to (3.1). Clearly $A = A_1 \otimes A_2 \cdots \otimes A_u \subseteq N(J_0)$ and by consequence $I(A) \supseteq IN(J_0) \supseteq J_0$ and $\text{codim } I(A) = \dim A = r_1 \dots r_u = \text{codim } J_0$. It follows that $J_0 = IN(J_0)$ and $A = N(J_0)$.

Now $N(J) = \{x \in A \mid l(x) = 0 \text{ for all } l \in \mathfrak{g}_A(J/J_0)\}$ and its dimension must be equal to $\dim A - \dim J/J_0$. Hence $\text{codim } IN(J) = \dim N(J) = \dim A - \dim J/J_0 = \text{codim } J_0 - \dim J/J_0 = \text{codim } J$. Thus $J = IN(J)$.

(4.4) Corollary. *The set of invariant subspaces of $C(\mathbb{Z}_p^u \rightarrow K)$, ordered by inclusion, satisfies the descending chain condition (i.e. every descending chain of invariant subspaces is finite.).*

Proof. This follows from (4.3) and the ascending chain property for ideals in $K\langle X_1, \dots, X_u \rangle$.

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